Rice University Economics Math Camp 2019
Topology Notes

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Contents

*These notes are compiled from a variety of sources, including but certainly not limited to Dugandgi's *Topology*; Aliprantis and Border's *Infinite Dimensional Analysis-A Hitchhiker's Guide*; Gamlin and Greene's *Introduction to Topology*; and notes from Professor Stephen Semmes. All errors are, of course, my own.

Preface

These notes were prepared primarily for my use in teaching. Therefore, they primarily include definitions, propositions, and some proofs, and do not include much explanation or motivation. Moreover, they only include proofs which I either presented or would have presented if time permitted.

Part I

Topology and Metric Spaces

1 Introduction to Topology

1.1 Definition of a topology and examples

Definition 1. Let X be a set. A collection τ of subsets of X is a **topology** if:

1. $\varnothing, X \in \tau$
2. If $U_1, ..., U_n \in \tau$, then $\cap_{i=1}^n U_i \in \tau$.
3. Let A be nonempty and let $U_\alpha \in \tau$ for all $\alpha \in A$. Then $\cup_{\alpha \in A} U_\alpha \in \tau$.

Note that this is not the same as the definition of a $\sigma-$algebra that you were exposed to in stat camp! To contrast them, I will provide you with the definition of a $\sigma-$algebra.

Definition 2. Let X be a set. A collection Σ of subsets of X is a **sigma algebra** if:

1. $\varnothing, X \in \Sigma$
2. If $E \in \Sigma$, then $E^C \in \Sigma$.
3. Let $E_n \in \Sigma$ for all $n \in \mathbb{N}$. Then $\cup_1^\infty E_i \in \Sigma$.

Elements of τ are called **open sets**. A **topological space** is denoted as (X, τ). When the topology is clear from context, sometimes just X will be used.

Examples:

1. For any X, the **indiscrete topology** on X is $\{\varnothing, X\}$.
2. For any X, the **discrete topology** on X is $\mathcal{P}(X)$, the power set of X.
3. Let $X = \mathbb{R}$. The **standard topology** on $\mathbb{R}$ is such that a subset $U \subseteq \mathbb{R}$ is said to be open if $\forall\, x \in U$, there exists an $a, b \in \mathbb{R}$ such that $a < x < b$ and $(a, b) \subseteq U$.
4. Let $X = \mathbb{R}$, and consider the topology consisting of subsets of X such that, for all $x \in U$, $\exists b \in \mathbb{R}$ such that $x < b$ and $[x, b) \subseteq U$. Denote this topology by τ_+.
5. Let $X = \mathbb{R}$, and consider the topology consisting of subsets of X such that, for all $x \in U$, $\exists a \in \mathbb{R}$ such that $x > a$ and $(a, x] \subseteq U$. Denote this topology by τ_-.
6. For any X, the **cofinite topology** is $\{U \subseteq X | X \setminus U \text{ is finite } \vee\ U = \varnothing\}$.

How do we show that something is a topology? I will demonstrate this the cofinite topology. Before I do so, however, recall the following definitions:

Definition 3. A set X is finite if it is the empty set or if there is a bijection $f : \{1, ..., n\} \to X$.

Definition 4. A set X is infinite if it is not finite.

Definition 5. A set X is countable if there is a bijection $f : \mathbb{Z}_{++} \to X$.

Definition 6. A set X is uncountable if it is infinite and not countable.

Also, recall DeMorgan's Laws. Let A be a nonempty set, then:

1. $(\cap_{\alpha \in A} B_\alpha)^C = \cup_{\alpha \in A} B_\alpha^C$.
2. $(\cup_{\alpha \in A} B_\alpha)^C = \cap_{\alpha \in A} B_\alpha^C$.

Claim 1. The cofinite topology is a topology

Proof. To see that the cofinite topology, τ, is a topology, note that it satisfies the three conditions of a topology.

1. The empty set is in it by definition, and $X^C = \varnothing$.
2. Let $U_1, U_2, ..., U_n \in \tau$. Then $(\cap_1^n U_i)^C = \cup_1^n U_i^C$. As the finite union of finite sets is finite, $(\cap_1^n U_i)^C$ is finite, so $\cap_1^n U_i \in \tau$.

3. Let A be nonempty and let $U_\alpha \in \tau$ for all $\alpha \in A$. Then $(\cup_{\alpha \in A} U_\alpha)^C = \cap_{\alpha \in A} U_\alpha^C$, which is finite, so $\cup_{\alpha \in A} U_\alpha \in \tau$.

$\square$

Another type of topological space is one given by a metric. A metric space is a topological space because a topology can be defined using a metric.

Recall the following definition:

Definition 7. Let X be a set. A function $d : X \times X \to \mathbb{R}_+$ is a **metric** if:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

A space (X, d) is a **metric space**.

How do we get a topology from a metric?

If $x \in X$ and $\varepsilon > 0$, then let $B_\varepsilon(x) = \{y \in X | d(x, y) < \varepsilon\}$. Define a set $U \subseteq X$ to be open with respect to $d(\cdot, \cdot)$ if for all $x \in U$, there exists a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. This defines a topology.

Examples:

1. The Euclidean metric on $\mathbb{R}$ gives the standard topology on $\mathbb{R}$.
2. The discrete metric gives the discrete topology.

Definition 8. A topological space (X, τ) is **metrizable** if τ is given by a metric.

Example:

- For any x, the discrete topology is induced from the discrete metric.
- For $X = \mathbb{R}$, the standard topology is induced from the Euclidean metric.

1.2 Topological properties of sets

Let (X, τ) be a topological space. A subset E of X is a **closed set** (with respect to τ) if $X \setminus E = \{x \in X | x \notin E\}$ is an open set in X (i.e. $X \setminus E$ is in τ).

Observe the following:

1. $\varnothing, X$ are closed.
2. If $E_1, ..., E_n$ are closed sets, $\cup_{i=1}^n E_i$ is closed.

3. Let A be nonempty and let E_α be closed for all $\alpha \in A$. Then $\cap_{\alpha \in A} E_\alpha$ is closed.

The above properties follow from DeMorgan's laws.

Definition 9. Let F be a subset of a topological space (X, τ). A point $x \in X$ is a **limit point** of F if for every open set $U \subseteq X$, with $x \in U$, there exists a $y \in F$ such that $y \in U$ and $x \neq y$.

Note: This is the same as the definition of accumulation point in the videos. Note also that the definition of an accumulation point across sources, with some requiring that an accumulation point have infinitely many points from F.

Definition 10. Let F be a subset of a topological space (X, τ). A point $x \in X$ is **adherent to** F if for every open set $U \subseteq X$ with $x \in U$, there exists a $y \in F$ such that $y \in U$.

Note: If either $x \in F$ or x is a limit point of F, then x is adherent to F.

Example. Let $X = \mathbb{R}$ with the standard topology, and let $F = (0, 1) \cup \{2\}$. The set of adherent points is $[0, 1] \cup \{2\}$, while the set of limit points is $[0, 1]$.

Definition 11. Let E be a subset of a topological space (X, τ). The **closure** of E, denoted by $\bar{E}$ or $cl(E)$, is the set of all $x \in X$ such that x is adherent to E.

Proposition 1. *If $E \subseteq X$ is closed, then $\bar{E} \subseteq E$.*

Proof. [Note: This is a proof by contrapositive, meaning that I show that if $x \notin E$, then $x \notin \bar{E}$.] Let $x \in X \setminus E$ be given. E is closed, which implies that $U = X \setminus E$ is open. This implies that $x \in U$ and $U \cap E = \varnothing$, so $x \notin \bar{E}$. □

Proposition 2. *If E is any subset of X, then $\bar{E}$ is a closed set. [Note: This was given on the homework]*

Proof. If $X \setminus \bar{E}$ is empty, then $\bar{E} = X$ which must be closed. Otherwise, for each $x \in X \setminus \bar{E}$, there exists an open set $U(x)$ such that $U(x) \subseteq X \setminus \bar{E}$. To see this, note that given the definition of the closure, there exists a set $U(x)$ such that $U(x) \subseteq X \setminus E$. If $U(x)$ contains an element of $\bar{E}$, then since this would be an open set containing a $z \in \bar{E}$, it must also contain an element of E, which would be a contradiction. Therefore, $X \setminus \bar{E} = \cup_{x \in X \setminus E} U(x)$, which is open, so $\bar{E}$ is closed. □

Definition 12. A point $x \in X$ is in the **boundary** of E, denoted by ∂E, if it is in both $\bar{E}$ and $\overline{X \setminus E}$.

Proposition 3. *The boundary of E is closed.*

Proof. The boundary of $E = \bar{E} \cap \overline{X \setminus E}$. As this is the finite intersection of closed sets, it is closed. □

Definition 13. The **interior** of a set E, denoted by E^O, is the set of all $x \in E$ such that there exists an open set $U(x)$ such that $x \in U(x)$ and $U(x) \subseteq E$.

Proposition 4. *The interior of any set E is an open set.*

Proof. Let $x \in E^O$. Then for every $x \in E^O$, there exists an open set $U(x)$ such that $x \in U(x)$. Let $U = \cup_{x \in E^O} U(x)$. Clearly U is an open set, so what remains to be shown is that $U = E^O$. To show equivalence, it is necessary to show that $E^O \subseteq U$ and $U \subseteq E^O$. First, note that it is clear that $E^O \subseteq U$, because for each $x \in E^O$, $x \in U(x)$ for some $U(x) \subseteq U$.
To show that $U \subseteq E^O$, note that if $U \nsubseteq E^O$, then there exists an $x \in E^O$ such that there is a $y \in U(x)$ such that $y \notin E^O$. As $U(x)$ is a subset of E for each $x \in X$, U is a subset of E, so it must be the case that $y \in E \setminus E^O$. This implies that $y \in E$ and there does not exist an open set $U(y)$ such that $y \in U(y)$ and $U(y) \subseteq E$. It is known, however, that $y \in U(x)$, which is an open set that is a subset of E, so this is a contradiction. □

Corollary. *The interior of an open set is itself.*

Definition 14. Let (X, τ) be a topological space. A subset E of X is said to be **dense** in X if for every $x \in X$ and open set $U \subseteq X$ with $x \in U$, there exists a $y \in E$ such that $y \in U$.

Examples:

1. If (X, τ) is any topological space, X is dense in itself.

2. If X is a nonempty set with the indiscrete topology, $E \subseteq X$ is dense if and only if $E \neq \varnothing$.

3. If X is a nonempty set with the discrete topology, $E \subseteq X$ is dense if and only if $E = X$.

4. A set $E \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ with respect to the standard topology if and only if for all $a, b \in \mathbb{R}$ with $a < b$, there exists an $x \in E$ such that $a < x < b$.

    - In particular, $\mathbb{Q}$ is dense in $\mathbb{R}$. To see this, note that $b - a > 0$. Therefore, there exists an $n \in \mathbb{Z}_{++}$ such that $n(b-a) > 1$. Let m be an integer such that $m - 1 \leq na < m$. Then, we have that $nb > na + 1 > m > na$, so $b > \frac{m}{n} > a$.

2 Topological Spaces

2.1 Separation Conditions

At this point, it is helpful to discuss some properties of topological spaces. Topology is fundamentally a way of defining how "close" objects in a space are, and the flip side of that is a way of distinguishing between objects that are "different". The separation conditions below fundamentally deal with the question of "In which topologies can objects of a particular type be separated by open sets?" Note that these are properties of the spaces themselves.

There are many different separation axioms, but for the purpose of this course I will define two of them:

1. A topological space (X, τ) is a T_1-space if for each pair of distinct points $x, y \in X$, there exists open sets U and V such that $x \in U$ and $y \notin U$, and such that $y \in V$ and $x \notin V$.

2. A topological space (X, τ) is a T_2-space (more commonly called a **Hausdorff space**), if for each pair of distinct points $x, y \in X$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

While these conditions look similar, there are subtle differences between them.

Examples:

1. A space that is not T_1: Any space X with at least two elements that has the indiscrete topology. For any y, the only open set containing y also contains $x \neq y$.

2. A space that is T_1 but not Hausdorff: Any infinite space X with the cofinite topology. Let $x \neq y$. Note that $X \setminus \{x\}$ is open in the cofinite topology, so (X, τ) satisfies T_1. However, for any open set U with $x \in U$, U^C is a finite set which is not open. As any subset of U^C is also finite and therefore not open, one cannot find an open set V such that $y \in U^C$ and $y \in V$.

Proposition 5. *(X, τ) satisfies T_1 if and only if for all $x \in X$, $\{x\}$ is closed.*

Proof. If (X, τ) has less than two elements, this is vacuously true, so assume that X has at least two elements.
If $\{x\}$ is closed for all $x \in X$, then $X \setminus \{x\}$ is open. This implies that for all $y \neq x$, the set $V = X \setminus \{x\}$ satisfies the required condition.
Suppose that (X, τ) satisfies T_1. Then for each $y \neq x$, there exists an open set $V(y)$ such that $x \notin V(y)$. This implies that $\cup_{y \neq x} V(y) = X \setminus \{x\}$ is open, so $\{x\}$ is closed. $\square$

Also note that it is clear that $T_2 \implies T_1$.

2.2 Subspaces

Definition 15. Let (X, τ) be a topological space, and let $Y \subseteq X$. The topology induced on Y by X is the topology such that a set $V \subseteq Y$ is open in Y if there exists an open set $U \subseteq X$ such that $V = U \cap Y$. If V is open in the induced topology, it is said to be **relatively open** in Y.

Note that if Y is open in X, then all sets that are relatively open in Y are open in X. Otherwise, this may not be true. For example, if $X = \mathbb{R}$ with the standard topology, and $Y = [0, 1]$, the set $A = [0, 0.5)$ is relatively open in Y.

Definition 16. Let (X, τ) be a topological space, and let $Y \subseteq X$. If E is a subset of Y and $E = Y \setminus V$ for a **relatively open** set V, then E is **relatively closed**.

Other types of sets are defined similarly. For example, relatively compact sets. It follows from this that, in a metric space, the restriction of $d(\cdot, \cdot)$ to Y defines a metric on Y, and this metric is the same as that induced from the topology on X given by that metric.

Proposition 6. *If (X, τ) satisfies T_1 (T_2) and $Y \subseteq X$, the topology induced by X on Y also satisfies T_1 (T_2).*

I prove this proposition below for the T_1 case. The other case is similar.

Proof. Let (X, τ) satisfy T_1. Then since $Y \subseteq X$, for every $x \in Y$, there exists an open set $U \subseteq X$ containing y such that $x \notin U$. As U is open in X, $V = U \cap Y$ is relatively open in Y, and $x \notin V$. Therefore, Y satisfies T_1. The result for T_2 is analogous. □

2.3 Connectedness

Definition 17. A topological space (X, τ) is **connected** if it fulfills any of the following three equivalent conditions:

1. There do **not** exist nonempty disjoint open sets $U_1, U_2 \subseteq X$ such that $X = U_1 \cup U_2$.
2. There do **not** exist nonempty disjoint closed sets $E_1, E_2 \subseteq X$ such that $X = E_1 \cup E_2$.
3. There are **no** sets other than X, $\varnothing$ that are both open and closed.

Examples:

1. If X has the indiscrete topology, it is connected.
2. If X has the discrete topology and at least two elements, it is not connected.
3. $\mathbb{R}$ with the standard topology is connected
4. $[0, 1] \cup [2, 3]$ with the topology induced by the standard topology on $\mathbb{R}$ is not connected

A set $A \subseteq X$ is connected if it is a connected subspace in the induced topology.

2.4 Bases

Definition 18. Let (X, τ) be a topological space. A family $\mathcal{B}$ is called a **base** (or basis) for a τ if each $U \in \tau$ can be expressed as the union of elements in $\mathcal{B}$.

The following is an equivalent way of defining a base:

Definition 19. Let (X, τ) be a topological space. A family $\mathcal{B}$ is called a **base** (or basis) for a τ if for each $U \in \tau$ and each $x \in U$, there is a $V \in \mathcal{B}$ such that $x \in V \subseteq U$.

I will call the first definition B1 and the second B2.

Definition 20. Proof of equivalence:

- B1 implies B2: Let $x \in U$. As $U \in \tau$ and $\mathcal{B}$ satisfies B1, $U = \cup_{\alpha \in A} V_\alpha$, where $V_\alpha \in \mathcal{B}$ for each α, so there exists a V_α such that $x \in V_\alpha$.
- B2 implies B1: Let $U \in \tau$. For each $x \in U$, there exists a $V(x) \subseteq U$, so $U = \cup_{x \in U} V(x)$.

Examples:

1. For any space X, τ is a base for itself.
2. For X with the discrete topology, the $\mathcal{B} = \{\{x\} | x \in X\}$ is a base for τ.
3. Let $X = \mathbb{R}$ with the standard topology, then the following are bases for τ:
   (a) $\mathcal{B} = \{(a, b) | a, b \in \mathbb{R}\}$
   (b) $\mathcal{B} = \{(a, b) | a, b \in \mathbb{Q}\}$
   This base is an important one, since it implies that X with the standard topology has a countable base. To see that this is true, it is easiest to work with definition B2. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, for any $x \in (a, b)$, there exists a $p, q \in \mathbb{Q}$ such that $a \leq p < x < q \leq b$, so $x \in (p, q) \subseteq (a, b)$.
4. Let X be a topological space and τ be a topology given by a metric. Then if E is dense in X, the set of open balls $\{B_{\frac{1}{j}}(x) | j \in \mathbb{Z}_{++}, x \in E\}$ forms a base for τ.

Definition 21. A topological space (X, τ) is **second countable** if it has a countable base.

Definition 22. Let (X, τ) be a topological space. A **local base** (or basis) for a topology τ at point $x \in X$ is a family of open sets $\mathcal{B}(x)$ such that $x \in V$ for every $V \in \mathcal{B}(x)$, and for every U in τ with $x \in U$, there exists a $V \in \mathcal{B}(x)$ such that $V \subseteq U$.

This is the "local" version of a base. For any of the examples above, a local base can be formed by taking the sets containing x. Moreover, an alternative definition of a base is a collection of local bases for every $x \in X$.

Definition 23. A topological space (X, τ) satisfies the **local countability condition** at x if there is a countable local base $\mathcal{B}(x)$ for x.

Definition 24. A topological space (X, τ) is **first countable** if every point x has a countable local base.

Note that any second countable space is first countable, but the reverse isn't true. An example of a first countable space that is not second countable is any uncountable set X with the discrete topology.

Proposition 7. *Every metric space is first countable.*

This is true because, for any $x \in X$, the set of open balls $\mathcal{B}(x) = \{B_{\frac{1}{n}}(x) | n \in \mathbb{Z}_{++}\}$ forms a local base for τ at x.

2.5 Comparing Topologies

Definition 25. Let X be a topological space and τ_1, τ_2 be topologies on X. Then if $\tau_1 \subseteq \tau_2$, i.e. for each open set $U \in \tau_1$, it is the case that $U \in \tau_2$, one says that τ_1 is coarser than τ_2, or equivalently that τ_2 is finer than τ_1. One can equivalently say that τ_1 is smaller than τ_2.

Examples:

1. The indiscrete topology on X is coarser than every other topology on X.

2. The discrete topology is finer than every other topology on X.

Proposition 8. *If $\tau_1 \subseteq \tau_2$, and (X, τ_1) is Hausdorff, then (X, τ_2) is Hausdorff.*

Proof. If (X, τ_1) is Hausdorff, then for each $x, y \in X$ there exist open sets $U, V \in \tau_1$ such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$. As these sets are also in τ_2, it must be the case that τ_2 is Hausdorff. □

Note that this is true for T_1 as well.

2.6 Product Spaces

Definition 26. Let $(X_1, \tau_1) \times ... \times (X_n, \tau_n)$ be topological spaces. The **product topology** on $X = X_1 \times ... \times X_n$ is the topology for which $W \subseteq X$ is open if for every $x \in W$, there exist open sets $U_1, U_2, ..., U_n$ such that $x_1 \in U_1$,...,$x_n \in U_n$, and $U_1 \times ... \times U_n \subseteq W$.

Equivalently, the product topology is given by a base consisting of the sets $\{U_1 \times ... \times U_n | U_j \in \tau_j\}$.

Claim 2. Let $(X_1, d_1(\cdot,\cdot)), ..., (X_n, d_n(\cdot,\cdot))$ be metric spaces and $X = X_1 \times ... \times X_n$. Then the metric $\max\{d_1(\cdot,\cdot), ..., d_n(\cdot,\cdot)\}$ determines the product topology on X. [The proof of this is part of the homework]

Note that the product topology on $X \times Y$ is not equal to the set $\tilde{\tau} = \{A \times B | A \in \tau_X, B \in \tau_Y\}$. To see why, consider the following counterexample:

Let $X = \{a, b\}$, $Y = \{1, 2\}$, $\tau_X = \{\emptyset, X, \{a\}\}$, and $\tau_Y = \{\emptyset, Y, \{1\}, \{2\}\}$. Then $\tilde{\tau} = \{\emptyset, X \times Y, X \times \{1\}, X \times \{2\}, \{a\} \times Y, \{a\} \times \{1\}, \{a\} \times \{2\}\}$. Note that $\{a\} \times Y$ and $X \times \{1\}$ are in $\tilde{\tau}$, but their union $\{a\} \times Y \cup X \times \{1\} = \{(a,1), (a,2), (b,1)\}$ is not in $\tilde{\tau}$.

The problem here is that $\tilde{\tau}$ does not contain enough sets.

3 Sequences and Compactness

3.1 Sequences

Definition 27. Let X be a set. A **sequence** is a function $f : \mathbb{Z}_{++} \to X$.

Definition 28. Let $\{x_n\}$ be a sequence in X and $\{n_k\}$ be a strictly increasing sequence in $\mathbb{Z}_{++}$. Then $\{x_{n_k}\}$ is a **subsequence** of $\{x_n\}$.

Definition 29. Let $\{x_n\}$ be a sequence in (X, τ) and x be an element of X. The sequence $\{x_n\}$ **converges** to x if for every open set U containing x, there exists a $N \in \mathbb{Z}_{++}$ such that, for all $n > N$, $x_n \in U$. If it does not converge to any $x \in X$, the sequence **diverges**.

In the case of a metric space, this definition is equivalent to the standard definition:

Definition 30. Let $\{x_n\}$ be a sequence in a metric space (X, d) and x be an element of X. The sequence $\{x_n\}$ **converges** to x if for every $\varepsilon > 0$, there exists a $N \in \mathbb{Z}_{++}$ such that, for all $n > N$, $d(x_n, x) < \varepsilon$.

In a general topological space, is the limit, if it exists, necessarily unique?

No, it is not. Let X be a set with at least two elements equipped with the indiscrete topology. Then each $x \in X$ is a limit of every sequence!

We know that the limit of a sequence in a metric space is unique, so there must be some condition on the topology that guarantees uniqueness of a limit. A logical place to look for such a condition is in the separation conditions (T_1, T_2,...), because those distinguish between points using open sets.

Is it sufficient for the space to be T_1?

No. This isn't sufficient either. For example, let $X = \mathbb{Z}_{++}$ and τ be the cofinite topology on X. Then let $\{x_n\}$ be a sequence such that $x_k = k$ for all $n \in \mathbb{Z}_{++}$. If $x \in X$, then x is a limit of $\{x_k\}$. This is true because each open set containing x excludes at most finitely many points. For any open set U, let N be the maximal excluded element. Then for all $k > N$, $x_k \in U$.

It turns out that, for a space with more than one element, a necessary and sufficient for every convergent sequence to have a unique limit is for it to be Hausdorff. I will show the sufficiency part here.

Proposition 9. *Let (X, τ) be a Hausdorff space and let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then x is unique.*

Proof. Let $\{x_n\}$ be a sequence in (X, τ) and assume that there exists an x, y with $x \neq y$ such that $\{x_n\}$ converges to x and $\{x_n\}$ converges to y. As (X, τ) is Hausdorff, it must be the case that there exist sets U and V with $x \in U$, $y \in V$ such that $U \cap V = \varnothing$. As $\{x_n\}$ converges to x, there exists a $N \in \mathbb{Z}_{++}$ such that for all $n > N$, $x_n \in U$. This implies, however, that for all $n > N$, $x_n \notin V$, which contradicts the assumption that $\{x_n\}$ converges to y. □

This leads us to another way to classify closed sets.

Definition 31. Let (X, τ) be a topological space. A set $E \subseteq X$ is **sequentially closed** if for every sequence $\{x_n\} \in E$ and every $x \in X$ such that $x_n \to x$, it is the case that $x \in E$.

How does this definition fit in with the definition of a closed set?

Proposition 10. *In any topological space, a closed set is sequentially closed.*

Proof. Let E be a closed set and let $x_n \to x$ with $x_n \in E$ for all n. Suppose that $x \notin E$. As E is closed, the set $U = E^C$ is open. As $x \in U$ and $x_n \to x$, it must be the case that there is a N such that for all $n \geq N$, $x_n \in U$, which is a contradiction. □

Is this reverse true? Not always, and you will show that there is a counterexample as part of your homework. **But it is true for first countable spaces, and therefore for metric spaces.**

Therefore, when dealing with metric spaces one can use the definitions interchangeably.

Now, I turn to some properties of sequences in metric spaces:

Definition 32. Let (X, d) be a metric space. A sequence $\{x_n\}$ is said to be a Cauchy sequence if for every $\varepsilon > 0$, there exists a $N \in \mathbb{Z}_{++}$ such that for all $m, n > N$, $d(x_m, x_n) < \varepsilon$.

Definition 33. Let (X, d) be a metric space. If every Cauchy sequence $\{x_n\}$ in X converges to an $x \in X$, then X is a **complete** metric space.

Examples:

1. $\mathbb{R}$ with the Euclidean metric is complete.

2. $(0, 1]$ with the Euclidean metric is not complete.
   To see this, note that $\{x_n\}$ where $x_n = \frac{1}{n}$ for each $n \in \mathbb{Z}_{++}$ is Cauchy, but its limit is zero which is not in $(0, 1]$.

Proposition 11. *Let (X, d) be a metric space, and $\{x_n\}$ be a convergent sequence in X. Then $\{x_n\}$ is Cauchy.*

(This was proven on the video, so I will not repeat it.)

Proposition 12. *Let (X, d) be a complete metric space. Then any closed subspace Y of X is complete (with respect to the metric induced from X).*

Proof. (X, d) is complete, so since $Y \subseteq X$, any sequence $\{x_n\}$ in Y converges to a $x \in X$. Because Y is closed, it is sequentially closed, so x must be in Y. □

3.2 Compact Sets

Definition. Let (X, τ) be a topological space. An **open cover** of a set $K \subseteq X$ is a family $\{U_\alpha\}_{\alpha \in A}$, where A is some set of indices, of open subsets of X such that $K \subseteq \cup_{\alpha \in A} U_\alpha$.

Definition. A set $K \subseteq X$ is said to be compact if every open cover of K in X can be reduced to a finite subcover. More precisely, this means that there are finitely many indices $\alpha_1, ..., \alpha_n \in A$ such that $K \subseteq \cup_{i=1}^n U_{\alpha_i}$.

Examples:

1. If X is any topological space and $K \subseteq X$ has only finitely many elements, then K is compact.
   This is true because for each $x \in K$, one can choose a $U(x) \in \{U_\alpha\}_{\alpha \in A}$ such that $x \in U(x)$, and $\cup_{x \in K} U(x)$ is a finite subcover of K.

2. If X is any set with the discrete topology and $K \subseteq X$ is compact, then K is finite.

   Proof. In the discrete topology, every set is open, so $\cup_{x \in K}\{x\}$ form an open cover of K. This only has a finite subcover if K is finite. □

3. If X is any set equipped with the indiscrete topology, then any subset of X is compact.
   To see this, note that the only possible open cover of K is $\{X\}$, which has only one element.

4. If $X = \mathbb{R}$ with the standard topology, then any closed interval $[a, b]$ is compact.

Proof. Suppose not. Then there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ such that there is no finite subcover. Let $U(a) \in \{U_\alpha\}_{\alpha\in A}$ be an open set containing a. This implies that there exists a $\varepsilon > 0$ such that $y \in U(a)$ for all $a < y < a+\varepsilon$, so clearly $[a, a+\varepsilon)$ can be covered by a finite subcover. If $[a, b]$ is not compact, then there must exist some $c \in [a, b]$ that is the largest element that can be covered by a finite subcover. Denote this subcover by $\{U_\alpha\}_{\alpha\in A^F}$, where A^F is a finite set. Now pick an open set $U(c) \in \{U_\alpha\}_{\alpha\in A}$. Note that as $U(c)$ is open, there must exist a $\varepsilon' > 0$ such that $y \in U(c)$ for all $c < y < c+\varepsilon'$. Note that $\{U_\alpha\}_{\alpha\in A^F} \cup U(c)$ is also finite, and it contains elements larger than c, which is a contradiction. □

5. Let X be any set with the cofinite topology and let K be any subset of X, then K is compact.

   Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be any open covering of K in X. If K is the empty set, then this is trivial. If $K \neq \varnothing$, then there is an $k_0 \in K$ such that $X \setminus U_{k_0}$ has only finitely many elements. In particular, $K \setminus U_{k_0}$ has only finitely many elements, so there are finitely many indices $k_1, ..., k_n$ such that $K \setminus U_{k_0} \subseteq \cup_1^n U_{k_j}$. Therefore, this together with U_{k_0} is a finite subcover of K. □

Note that, when $X = \mathbb{R}$ and X is equipped with the standard topology, an open interval (a, b) is not compact. To see this, consider the set $\cup_{i=1}^{\infty}(a + \frac{b-a}{2^{n+1}}, b - \frac{b-a}{2^{n+1}})$. This forms an open cover of (a, b) without a finite subcover.

Proposition 13. *Let X be a space and τ_1, τ_2 be topologies on X, with $\tau_1 \subseteq \tau_2$. If K is compact in (X, τ_2), then it is also compact in (X, τ_1).*

Proof. If K is compact in (X, τ_2), then for every open cover $\{U_\alpha\}_{\alpha\in A}$ in τ_2 that covers K, there exists a finite subcover $\{U_\alpha\}_{\alpha\in A^F}$, where A^F is a finite subset of A. As all of the open sets in τ_1 are also in τ_2, this applies for every open cover consisting of sets from τ_1. □

Proposition 14. *If $K \subseteq X$ is compact and $E \subseteq X$ is closed, then $K \cap E$ is compact.*

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of $K \cap E$. As E is closed, $X \setminus E$ is open, so $\{U_\alpha\}_{\alpha\in A} \cup X \setminus E$ is an open cover of K. As K is compact, this must have a finite subcover with at most finitely many U_α, so this forms a finite subcover for $K \cap E$. □

Proposition 15. *If X is Hausdorff and K is compact, then K is closed*

Proof. Let $x \in X \setminus K$ be given. For every $y \in K$, let $U(x, y)$ and $V(x, y)$ be disjoint open sets such that $x \in U(x, y)$ and $y \in V(x, y)$. (The Hausdorff assumption guarantees that these exist.) The collection of all sets $V(x, y)$, meaning $\{V(x, y)\}_{y\in K}$ is an open cover of K, so it has a finite subcover $V(x, y_1), ..., V(x, y_n)$ for some $y_1, ..., y_n \in K$. Let $V(x) = \cup_{i=1}^n V(x, y_i)$ and $U(x) = \cap_{i=1}^n U(x, y_i)$.

$U(x)$ is the finite intersection of open sets and is therefore open, and note that $U(x) \cap V(x) = \varnothing$. As x is arbitrary, this can be defined for any $x \in X \setminus K$, so $U = \cup_{x \in X \setminus K} U(x)$ is open and equal to $X \setminus K$, so K is closed. □

The Hausdorff assumption is required, because otherwise the cofinite topology would be a counterexample. Note that as all metric spaces are Hausdorff, this implies that in any metric space, a compact set is closed.

Definition 34. Let (X, τ) be a topological space. A set $K \subseteq X$ is sequentially compact if every sequence $\{x_n\} \in K$ has a subsequence $\{x_{n_k}\}$ that converges to an $x \in K$.

3.3 Metric-Space Properties

Definition 35. Let (X, d) be a metric space. A set $E \subseteq X$ is bounded if there exists an $x \in X$ and a $r \in \mathbb{R}$ such that $d(x, y) < r$ for all $y \in E$.

While on the topic of bounded sets, it would be a good time to define a bounded sequence. In $\mathbb{R}$, one can define a bound saying that $|x_n| < M$, where M is a real number, for all n. In a vector space, one can bound a sequence using the norm, i.e. that $||x_n|| \leq M$ for all n. In a general metric space, a sequence is bounded if it is in a bounded set.

Proposition 16. *Let $\{x_n\} \in X$ be a Cauchy sequence. Then $\{x_n\}$ is bounded.*

Proof. Let $\varepsilon = 1$, and note that, because $\{x_n\}$ is Cauchy, there exists an $N \in \mathbb{Z}_{++}$ such that for all $m, n > N$, $d(x_m, x_n) < 1$. This implies that for all $m > N$, $d(x_{N+1}, x_m) \leq 1$. Let $M = \max\{d(x_{N+1}, x_1), ..., d(x_{N+1}, x_N)\}$. Let $r = \max\{1, M\}$. Then for all $n \in \{x_n\}$, $d(x_{N+1}, x_n) \leq r$. □

Definition 36. Let (X, d) be a metric space. A set $E \subseteq X$ is totally bounded if for every $\varepsilon > 0$, there exists a finite set A_ε such that $E \subseteq \cup_{x \in A_\varepsilon} B_\varepsilon(x)$

Any totally bounded set is bounded, but the reverse is not true. For example if $X = \mathbb{R}$ with the discrete metric and $E = [0, 1]$, E is bounded but not totally bounded.

Proposition 17. *In a metric space (X, d), a set K is compact if and only if it is sequentially compact.*

In order to prove this proposition, I introduce the following lemma (whose proof is part of the homework)

Lemma 1. *Let (X, d) be a metric space and assume $K \subseteq X$ is a sequentially compact set. Then K is complete, closed, and totally bounded.*

of the proposition. First, assume that K is compact and let $\{x_n\}$ be a sequence in K. Let $S = \{x_n | n \in \mathbb{Z}_{++}\}$. If S is finite, it must have a constant subsequence, which must converge in K, so from now on assume that S is infinite. Note that, because X is a metric space, K is closed, so $\bar{S} \subseteq K$.

Moreover, as I show below, there exists an $a \in \bar{S}$ such that, for all $\varepsilon > 0$, there are infinitely many elements of S in $B_\varepsilon(a)$. Suppose not. Then for each $a \in \bar{S}$, there exists a $\delta_a > 0$ such that $B_{\delta_a}(a)$ has only finitely many elements. Let $d_a = \min_{b \neq a, b \in \in B_{\delta_a(a)}}\{d(a,b)\}$, and let $\varepsilon_a < d_a$. Then $B_{\varepsilon_a}(a)$ is the singleton set $\{a\}$. The collection $\{B_{\varepsilon_a}(a)\}_{a \in \bar{S}}$ of all such $\varepsilon-$balls, together with $\bar{S}^C$, forms an infinite open cover of K without a finite subcover, which is a contradiction.

Therefore, there is a subsequence converging to $a \in K$, and K is sequentially compact.

Now assume that K is sequentially compact, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of K. Assume that there does not exist a finite subcover. Let $\varepsilon = \sup_{x,y \in K} d(x,y)$. Using the lemma, there exist finitely many **closed** balls of radius $\frac{\varepsilon}{4}$ that cover K. Given our assumption that $\{U_\alpha\}_{\alpha \in A}$ does not have a finite subcover, there must exist an $x_1 \in K$ such that $K_1 = B_{\frac{\varepsilon}{4}}(x_1) \cap K$ does not have a finite subcover (meaning that there does not exist a finite set A^F such that $K_1 \subseteq \cup_{\alpha \in A^F} U_\alpha$).

Now consider the set K_1. Given that K_1 does not have a finite subcover, there must exist an $x_2 \in K_1$ such that $K_2 = B_{\frac{\varepsilon}{16}}(x_2) \cap K_1$ does not have a finite subcover. Similarly, for any K_i, one can find an x_{i+1} such that $K_{i+1} = B_{\frac{\varepsilon}{4^i}}(x_2) \cap K_i$ such that K_{i+1} does not have a finite subcover. This sequence $\{x_n\}$ forms a Cauchy sequence, which, due to the lemma, must converge in K. This implies that $\cap_{n \in \mathbb{Z}_{++}} K_n = \{x\}$. Let $U(x)$ be the element of U containing x, and there exists an N such that for all $n > N$, $K_n \subseteq U(x)$. This contradicts the assumption that K_n does not have a finite subcover, and this implies that K must have a finite subcover as well. □

Results for $\mathbb{R}^n$ with the Euclidean metric:

Theorem (Bolzano-Weierstrass)**.** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Proof. First, I show this for $\mathbb{R}$.

As $\{x_n\}$ in bounded, there exists an $[a,b]$ such that $x_n \in [a,b]$ for all n. Note that, as there are infinitely many terms of this sequence, it must be the case that $[a, \frac{a+b}{2}]$ or $[\frac{a+b}{2}, b]$ has infinitely many terms of the sequence. Call one such interval (they could both have infinitely many terms) $[a_1, b_1]$, and let x_{n_1} be a member of this interval. Then note that one can similarly divide $[a_1, b_1]$ into two halves, one with infinitely many terms, call this half $[a_2, b_2]$ and let x_{n_2} be a member of this interval (with $n_2 > n_1$). Note that these intervals become half the size at each iteration, so $d(a_n, b_n) \to 0$ as $n \to \infty$, so $\lim_{k\to\infty} a_k = \lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} b_k$, so $\{x_{n_k}\}$ converges.

Now suppose the sequence is in $\mathbb{R}^n$. First, one can find a subsequence that converges in the first dimension. Then from that sequence, one can find a subsequence that converges in the second dimension, etc. □

Theorem (Heine-Borel)**.** *Let $K \subseteq \mathbb{R}^n$, then K is compact if and only if it is closed and bounded.*

Proof. We have already established that in $\mathbb{R}^n$, any compact set K is closed and totally bounded, and therefore it is also bounded. All that remains to be shown is that any closed and bounded subset K of $\mathbb{R}^n$ is compact. The fact that K is bounded implies, by the Bolzano-Weierstrass Theorem, that every sequence $\{x_n\} \subseteq K$ has a convergent subsequence, and the fact that K is closed implies that this subsequence must converge within K. This implies that K is sequentially compact, and therefore compact. □

Before we move on to functions, I present another characterization of compactness. I do so because this property can be useful for proofs involving compactness.

Proposition 18. *Let (X, τ) be a topological space, then X is compact, meaning that the space itself is a compact set, if and only if for every collection of closed subsets $\{E_\alpha\}_{\alpha \in A}$ satisfying the FIP, the arbitrary intersection of elements of these sets is nonempty.*

Proof. Assume that X is compact and assume that there exists a family of closed subsets $\{E_\alpha\}_{\alpha \in A}$ that has the FIP but for which some arbitrary intersection of these sets is empty. This implies that there exists an infinite set $A' \subseteq A$ such that $\cap_{\alpha \in A'} E_\alpha = \varnothing$. For each $\alpha \in A'$, let $U_\alpha = E_\alpha^C$, and note that, because $\cap_{\alpha \in A'} E_\alpha = \varnothing$, the collection of sets $\{U_\alpha\}_{\alpha \in A'}$ forms an open cover of X. However, because $\{E_\alpha\}_{\alpha \in A}$ has the finite intersection property, it must be the case that for any finite subset $A^F \subseteq A'$, $\cap_{\alpha \in A^F} E_\alpha = E \neq \varnothing$, so $\cup_{\alpha \in A^F} U_\alpha \subsetneq X$, and therefore does not cover X. This means that $\{U_\alpha\}_{\alpha \in A'}$ has no finite subcover, which contradicts the assumption that X is compact.

Assume that X has the condition that for every collection of closed subsets $\{E_\alpha\}_{\alpha \in A}$ satisfying the FIP, the arbitrary intersection of elements of these sets is nonempty is true, and let $\{U_\alpha\}_{\alpha \in A'}$ be an open cover of X, meaning that $X = \cup_{\alpha \in A'} U_\alpha$. This implies that $\cap_{\alpha \in A'} U_\alpha^C = \cap_{\alpha \in A'} E_\alpha = \varnothing$. Given our assumption, it must be the case that there exists a finite set A^F such that $\cap_{\alpha \in A^F} E_\alpha = \varnothing$, which implies that $X = \cup_{\alpha \in A^F} U_\alpha$, so X has a finite subcover and X is compact. □

4 Functions and Correspondences

4.1 Functions

Definition 37. Let X and Y be non-empty sets. A **function** $f : X \to Y$ is a subset of $X \times Y$ such that for each $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in f$, with this being denoted as $f(x) = y$.

Definition 38. Let $f : X \to Y$ and $g : Y \to Z$. The composition $h = g \circ f : X \to Z$ is the function such that $h(x) = g(f(x))$.

Definition 39. Let $f : X \to Y$ be a function. The **inverse** of f is a function f^{-1} such that $f^{-1}(f(x)) = x$.

Definition 40. Let $f : X \to Y$ be a function. The **image** of a set $A \subseteq X$, $f(A)$, is equal to $\{y \in Y | \exists x \in A \text{ s.t. } y = f(x)\}$.

Definition 41. Let $f : X \to Y$ be a function, the **preimage** of a set $B \subseteq Y$, $f^{-1}(B)$ is equal to $\{x \in X | f(x) \in B\}$.

Definition 42. A function $f : X \to Y$ is **one-to-one** or **injective** if $f(x) = f(y)$ implies that $x = y$.

Definition 43. A function $f : X \to Y$ is **onto** or **surjective** if for $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$.

Definition 44. A function $f : X \to Y$ is **bijective** if it is both one-to-one and onto.

4.1.1 Continuous Functions

Definition 45. A function $f : X \to Y$ is **continuous at** x if for every open set $V \subseteq Y$ with $f(x) \in V$, there exists a $U \subseteq X$ such that $x \in U$ and $f(U) \subseteq V$. f is **continuous** if it is continuous at x for all $x \in X$.

Corollary. *If $f : X \to Y$ is continuous, then $f^{-1}(V)$ is open.*

To see why the corollary is true, note that for each $x \in f^{-1}(V)$, there exists a function $U(x)$ such that $x \in U(x)$ and $U(x) \subseteq f^{-1}(V)$. This implies that $\cup_{x \in f^{-1}(V)} U(x) = f^{-1}(V)$, and this is an open set.

Examples of continuous functions:

1. If $f : X \to Y$ and X has the discrete topology, then it is continuous.

2. If $f : X \to Y$ and Y has the indiscrete topology, then it is continuous.

3. If X and Y are metric spaces, then this definition is equivalent to the standard $\varepsilon-\delta$ definition: A function $f : (X, d) \to (Y, \rho)$ is continuous at x if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that whenever $d(x, y) < \delta$, $\rho(f(x), f(y)) < \varepsilon$.

4. If $f : X \to Y$ is constant, then it is continuous.

5. If X and Y have the same topology, the identity mapping is continuous.

Definition 46. A function $f : X \to Y$ is **sequentially continuous at** x if for every sequence $x_n \to x$, $f(x_n) \to f(x)$ in Y. It is sequentially continuous if it is sequentially continuous at x for all $x \in X$.

Proposition 19. *If $f : X \to Y$ is continuous at x, then it is sequentially continuous at x.*

Proof. Suppose not. Then there exists a sequence $\{x_n\} \in X$ such that $x_n \to x$ but $f(x_n) \not\to f(x)$. This implies that there exists a $V \subseteq Y$ such that for all $N \in \mathbb{Z}_{++}$, there exists an $n > N$ such that $f(x_n) \notin V$. As this is true for each $N \in \mathbb{Z}_{++}$, there are infinitely many such terms. As f is continuous, this implies that $U \subseteq f^{-1}(V)$ is open in X, but that for any N, one can find an $k > N$ such that $f(x_k) \notin V$, so $x_k \notin U$, which contradicts the assumption that $x_n \to x$. □

Is the converse true? As you will show on your homework, it is not true in general. However, the converse is true in spaces where X is first countable. For the purposes of this course, however, I will just prove this for the metric space case.

Proposition 20. *If X and Y are metric spaces and f is sequentially continuous, then f is continuous.*

Proof. To show this, I prove the contrapositive, meaning that I show that if f is not continuous, then it is not sequentially continuous. If f is not continuous, then there exists an $x \in X$ and an $\varepsilon > 0$, such that for all $\delta > 0$, there exists a $x_\delta \in X$ such that $d(x, x_\delta) < \delta$ and $\rho(f(x), f(x_\delta)) > \varepsilon$. As this is true for all δ, we have that $d(x, x_{\frac{1}{n}}) < \frac{1}{n}$ for all $n \in \mathbb{Z}_{++}$. Letting $x'_n = x_{\frac{1}{n}}$, we have that the sequence $x'_n \to x$, but $\rho(f(x), f(x'_n)) > \varepsilon$ for all n, so $f(x'_n) \not\to f(x)$, implying that f is not sequentially continuous. □

Proposition 21. *If $f : X \to Y$ is continuous and $K \subseteq X$ is compact, then $f(K)$ is compact.*

Proof. Let $K \subseteq X$ be compact, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $f(K)$. As f is continuous, $f^{-1}(\{U_\alpha\}_{\alpha \in A})$ is an open cover of K. As K is compact, there must exist a finite subset $A^F \subseteq A$ such that $f^{-1}(\{U_\alpha\}_{\alpha \in A^F})$ also covers K, so $\{U_\alpha\}_{\alpha \in A^F}$ is a finite subcover of $f(K)$. □

Proposition 22. *If $f : X \to Y$ is continuous and $A \subseteq X$ is connected, then $f(A)$ is connected.*

Proof. Let A be connected subset of X and suppose that $f(A)$ is not connected. This implies that there exist two disjoint open sets U and V such that $f(A) = U \cup V$. This implies that for each $x \in A$, either $f(x) \in U$ or $f(x) \in V$ (and not both). Let $A_1 = \{x \in A | f(x) \in U\} = f^{-1}(U)$ and $A_2 = \{x \in A | f(x) \in V\} = f^{-1}(V)$. Clearly, $A = A_1 \cup A_2$, and since U and V are disjoint, A_1 and A_2 are disjoint. As U and V are open, A_1 and A_2 are open as well, which implies that A is not connected, which is a contradiction. □

Definition 47. A **homeomorphism** between X and Y is a bijection f such that f and f^{-1} are both continuous. X and Y are **homeomorphic** is there exists a homeomorphism between them.

A homeomorphism preserves all topological properties (openness, closedness, compactness, connectedness, etc.)

Example 1. In the standard topology, $(0,1)$ and $\mathbb{R}$ are homeomorphic. Let $f : \mathbb{R} \to (0,1)$ be such that $f(x) = \frac{1}{1+e^{-x}}$, $f^{-1} : (0,1) \to \mathbb{R}$ is then such that $f(y) = -\ln(\frac{1}{x} - 1)$.

Proposition 23. *Let X be a compact topological space and Y be a Hausdorff space, then if f is a bijection between X and Y and f is continuous, then it is a homeomorphism.*

Proof. Let f be a continuous bijection between X and Y. This claim is true if f^{-1} is continuous, meaning that for every closed set $E \subseteq X$, $(f^{-1})^{-1}(E) = f(E)$ is closed. As X is compact and E is closed, E is also compact. This implies that $f(E)$ is also compact. Because Y is Hausdorff, this implies that $f(E)$ is also closed. $\square$

4.1.2 Properties of Extended-Real Valued Functions

This section deals specifically with some properties related to functions that take values on the extended real line, i.e. $\mathbb{R} \cup \{-\infty, +\infty\}$. Denote this as $\bar{\mathbb{R}}$. Define the standard topology on $\bar{\mathbb{R}}$ as such that a subset $U \subseteq \bar{\mathbb{R}}$ is said to be open if $\forall\, x \in U$, either:

- $x \in \mathbb{R}$ and there exists an $a, b \in \mathbb{R}$ such that $a < x < b$ and $(a, b) \subseteq U$,
- $x = -\infty$ and there exists an $b \in \mathbb{R}$ such that $[-\infty, b) \subseteq U$,
- $x = +\infty$ and there exists an $a \in \mathbb{R}$ such that $(a, +\infty] \subseteq U$.

In this section, $\bar{\mathbb{R}}$ will always be assumed to have the standard topology.

Definition 48. The **limit inferior** of an extended-real valued sequence is defined as $\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \inf_{m>n} x_m = \sup_{n\geq 1} \inf_{m\geq n} x_m$ and the **limit superior** of a sequence is defined as $\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup_{m>n} x_m = \inf_{n\geq 1} \sup_{m\geq n} x_m$.

Note that if one defines $E(\{x_n\})$ to be the set of all subsequential limits of a sequence $\{x_n\}$, then $\liminf_{n\to\infty} x_n = \inf(E(\{x_n\}))$ and $\limsup_{n\to\infty} x_n = \sup(E(\{x_n\}))$.

Definition 49. A function $f : X \to \bar{\mathbb{R}}$ is **upper semicontinuous** (USC) if for each $c \in \bar{\mathbb{R}}$, the set $\{x \in X | f(x) < c\}$ is open.

Definition 50. A function $f : X \to \bar{\mathbb{R}}$ is **lower semicontinuous** (LSC) if for each $c \in \bar{\mathbb{R}}$, the set $\{x \in X | f(x) > c\}$ is open.

Note that a real-valued function is continuous if and only if it is both lower and upper semicontinuous. If X is a first-countable topological space, the following definitions of semicontinuity are equivalent:

Definition 51. A function $f : X \to \bar{\mathbb{R}}$ is **upper semicontinuous** if $x_n \to x \implies \limsup_{n\to\infty} f(x_n) \leq f(x)$.

Definition 52. A function $f : X \to \bar{\mathbb{R}}$ is **lower semicontinuous** if $x_n \to x \implies \liminf_{n\to\infty} f(x_n) \geq f(x)$.

Proposition 24. *A USC (LSC) function $f : X \to \bar{\mathbb{R}}$, where X is compact, obtains its maximum (minimum), and the set of maximizers (minimizers) is compact.*

Proof. I will show this for the USC case, but the proof is the same (with the inequalities reversed) for the LSC case. Let X be a compact space and let $f : X \to \bar{\mathbb{R}}$ be a USC function. Let $A = f(X)$, and for each $a \in A$, let $F_a = \{x \in X : f(x) \geq a\}$. Since f is USC, F_a is closed. The family of sets $\{F_a | a \in A\}$ has the FIP. Because X is compact, $\cap_{a\in A} F_a$ is nonempty and compact, and this is the set of maximizers of f. □

4.2 Correspondences

Definition 53. Let X and Y be non-empty sets. A **correspondence** F, $F : X \twoheadrightarrow Y$, is a subset of X×Y, with $(x, y) \in F$. The set of y such that $(x, y) \in F$ is denoted by $F(x)$.

In particular, one should note that:

- $F(x)$ can be empty.
- If $F(x)$ is single-valued for all $x \in X$, then F is simply a function.
- F can also be viewed as a function mapping from X to $\mathcal{P}(Y)$.

Definition 54. A correspondence $F : X \twoheadrightarrow Y$ is *closed-valued* if $F(x)$ is a closed set for each $x \in X$ and it is *compact-valued* if $F(x)$ is a compact set for each $x \in X$.

Definition 55. The *graph* of a correspondence is the set $Gr(F) = \{(x, y) \in X \times Y | y \in F(x)\}$.

Now that the correspondences themselves have been defined, one can define the concept of an inverse for a correspondence. There are two different ways to define the inverse of a correspondence, which are given below:

Definition 56. Let $F : A \twoheadrightarrow B$. The **lower inverse** of F is the mapping $F^- : B \to A$ such that for any non-empty subset C of B, $F^-(C) = \{a \in A | F(a) \cap C \neq \varnothing\}$.

Definition 57. Let $G : A \twoheadrightarrow B$. The **upper inverse** of F is the mapping $F^+ : B \to A$ such that for any non-empty subset C of B, $F^+(C) = \{a \in A | F(a) \subseteq C\}$.

The difference between these two definitions is that in the first, we include any $a \in A$ such that there is at least one element of $F(a)$ in C, while the second definition requires that the entire $F(a)$ be in C. Therefore, it is evident that $F^+(C) \subseteq F^-(C)$.

In the online lectures the following definitions of semicontinuity (more commonly referred to as **hemicontinuity**) were provided:

Definition 58. Let (X, d) and (Y, ρ) be metric spaces. A correspondence $F : X \twoheadrightarrow Y$ is **upper semicontinuous (hemicontinuous) (USC/UHC)** at $x \in X$ if whenever $x_n \to x$, $y_n \in F(x_n)$, and $y_n \to y$, then $y \in F(x)$. A correspondence is upper semicontinuous if it is upper semicontinuous at each $x \in X$.

Definition 59. Let (X, d) and (Y, ρ) be metric spaces. A correspondence $F : X \twoheadrightarrow Y$ is **lower semicontinuous (hemicontinuous) (LSC/LHC)** at $x \in X$ if whenever $x_n \to x$ and $y \in F(x)$, there exists a $y_n \in F(x_n)$ such that $y_n \to y$. A correspondence is lower semicontinuous if it is lower semicontinuous at each $x \in X$.

Definition 60. A correspondence $G : X \to Y$ is **continuous** if it is both USC and LSC

For the remainder of the course, I will refer to these definitions as **sequential semicontinuity** (abbreviated SUSC and SLSC respectively).[1]

Definition 61. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A correspondence $F : X \twoheadrightarrow Y$ is **lower semicontinuous** at $x \in X$ if for each open set $V \subseteq Y$ such that $V \cap F(x) \neq \varnothing$, there exists an open set $U(x) \subseteq X$ such that for all $x' \in U(x)$, $F(x') \cap V \neq \varnothing$.

For the purpose of this course, I will call this **"Open-set lower semicontinuity (OSLSC)"**.

There is an analogous definition for upper semicontinuity:

Definition 62. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A correspondence $F : X \twoheadrightarrow Y$ is **upper semicontinuous** at x if for each open set $V \subseteq Y$ such that $F(x) \subseteq V$, there exists an open set $U(x) \subseteq X$ such that for all $x' \in U(x)$, $F(x') \subseteq V$.

I refer to this as **"Open-set upper semicontinuity (OSUSC)"**

Examples:

Consider the following correspondences from $[0, 1] \to [0, 1]$, where $[0, 1]$ is endowed with the topology induced by the standard topology on $\mathbb{R}$.

Example 2. Let $F(x) = [\frac{1}{4}, \frac{3}{4}]$ for all $x < \frac{1}{2}$, and $F(x) = \frac{1}{2}$ for all $x \geq \frac{1}{2}$.

Example 3. Let $F(x) = [\frac{1}{4}, \frac{3}{4}]$ for all $x \leq \frac{1}{2}$, and $F(x) = \frac{1}{2}$ for all $x > \frac{1}{2}$.

The first correspondence is not USC but is LSC, while the second correspondence is LSC but not USC, and this is according to both definitions.

Proposition 25. *A necessary and sufficient condition for G to be OSLSC is that for each open set Z in Y, $G^{-}(Z)$ is open.*

[1] This is a term that I am using to refer to it. I'm not sure if anyone else uses that term.

Proof. Suppose that G is OSLSC. If $G^-(Z)$ is empty then it is open. Otherwise, let $x_0 \in G^-(Z)$. This implies that $G(x_0) \cap Z \neq \varnothing$, and so there is an open set $U(x_0)$ such that for all $x \in U(x_0)$, $G(x) \cap Z \neq \varnothing$, so $U(x_0) \subseteq G^-(z)$. Taking the union over each $x \in G^-(Z)$, it is evident that $G^-(Z)$ is open.

If $G^-(Z)$ is open for each $Z \in Y$, then for any Z such that $G(Z) \cap G(x_0) \neq \varnothing$, $G^-(Z)$ is an open set in X containing x_0 such that for all $x \in G^-(Z)$, $G(x) \cap G(Z)$ is nonempty, so G is OSLSC. □

Proposition 26. *A necessary and sufficient condition for G to be OSUSC is that for each open set Z in Y, $G^+(Z)$ is open.*

Proof. Suppose that G is OSUSC. If $G^+(Z) = \varnothing$ it is open. Otherwise, let $x_0 \in G^+(Z)$, then there exists an open set $U(x_0)$ such that for all $x \in U(x_0)$, $G(x)$ is open, so $U(x_0) \in G^+(Z)$. Therefore, taking the union over all such open sets, it is evident that $G^+(Z)$ is open.

If $G^+(Z)$ is open for each $Z \subseteq Y$, then let $x_0 \in X$ and Z be an open set containing $G(x_0)$. Then for all $x \in G^+(Z)$, $G(x) \subseteq Z$, so G is OSUSC. □

As you recall, in the online math camp you proved that, **in metric spaces**,[2] the two definitions of lower-semicontinuity are equivalent. For completeness, I present the proof of equivalence here:

Proposition 27. *Let X and Y be metric spaces and let $F : X \twoheadrightarrow Y$. F is OSLSC iff it is SLSC.*

Proof. Let F be OSLSC and let $x_n \to x$ and let y be an arbitrary element of $F(x)$. For any $\varepsilon > 0$, let $B_\varepsilon(y)$ be the ε-ball containing y. As F is OSLSC, there must be a $\delta > 0$ such that for all $x' \in B_\delta(x)$, $F(x') \cap V \neq \varnothing$. This implies that there exists an $N \in \mathbb{Z}_{++}$ such that, for all $n > N$, $F(x_n) \cap V \neq \varnothing$. Therefore, for each ε, one can choose a $y_n \in B_\varepsilon(y)$, and the sequence of these y_n terms converges to y, so it is SLSC.

Let F be SLSC, and suppose that F is not OSLSC. This implies that there exists an $x \in X$ with some $y \in F(x)$ and some $\varepsilon > 0$ such that $B_\varepsilon(y) \cap F(x) \neq \varnothing$, for which there is no $\delta > 0$ with $B_\delta(x) \subseteq X$ such that for all $x' \in B_\delta(x)$, $F(x') \cap V \neq \varnothing$. This implies that, for each $n \in \mathbb{Z}_{++}$, there exists an $x_n \in B_{\frac{1}{n}}(x)$ such that $F(x_n) \cap V \neq \varnothing$. This sequence converges to x, but there does not exist a $y_n \in F(x_n)$ that converges to y, contradicting the assumption that F is SLSC. □

Are the two definitions given for upper-semicontinuity equivalent in general? It turns out that they are not even equivalent in metric spaces!

Consider, for example, the following single-valued correspondence from $\mathbb{R}_+$ to $\mathbb{R}_+$ with the Euclidean metric:

$$F(x) = \begin{cases} \{\frac{1}{x}\} & x > 0, \\ 0 & x = 0. \end{cases}$$

[2] This equivalence holds more generally in first-countable topological spaces.

This correspondence satisfies the sequential definition (as there are no sequences in $F(x)$ that converge to 0, but does not satisfy the open set definition.

However, when the correspondence is **compact-valued and the range of F is contained in a compact subset $K \subseteq Y$,** the definitions are equivalent when X is first-countable and Y is **metrizable**. For the proof, assume that both X and Y are metric spaces.

Proof. Let F satisfy the open-set definition and be compact valued. Let $x_n \to x$ in X and let $\{y_n\}$ be a convergent sequence with $y_n \in F(x_n)$ for all n. Suppose that $y \notin F(x)$. Then this implies that for every $y' \in F(x)$, there exists an open set $V(y')$ and a $n_{y'} \in \mathbb{Z}_{++}$ such that for all $n > n_{y'}$, $y_n \notin V(y')$. Note that $\{V(y')\}_{y' \in F(x)}$ forms an open cover of $F(x)$, and since this is a compact set, there must exist a finite subcover $V(y_1), ..., V(y_n)$. Let $V = \cup_{i=1}^{m} V(y_i)$, and let $n = \max\{n_{y^1}, ..., n_{y^m}\}$. This implies that for all $k > n$, $y_n \notin V$. However, as F is open-set USC, it must be the case that for a large enough k, $F(x_k) \subseteq V$, which is a contradiction.

Let F be sequentially USC, meaning that every sequence $\{x_n, y_n\}$ in $Gr(F)$, if $x_n \to x$, then if y_n converges y, $y \in F(x)$. If F is not open-set USC at x, then there exists an open set V containing $F(x)$ such that there exists an $n \in \mathbb{Z}_{++}$ such that for all $m > n$, there exists an $x_m \in B_{\frac{1}{m}}(x)$ such that there is a $y_m \in F(x_m)$ and $y_m \notin V$. The sequence of all such x_m converges to x, but the limit of the sequence of y_m terms is not in $F(x)$. □

5 Applications

5.1 Maximum Theorems

Theorem (Weierstrass)**.** *Let (X, τ_X) be a compact topological space and let $\mathbb{R}$ be endowed with the standard topology. Then if $f : X \to \mathbb{R}$ is continuous, it obtains its maximum and minimum.*

Proof. I will show this for the maximum, as the proof for the minimum case is the same. As X is compact, $f(X) = K$ is a compact subset of $\mathbb{R}$. The Heine-Borel Theorem implies that K must be closed and bounded, so there exists a $y = \sup_x\{f(x)\} \in K$, which implies that there exists an x such that $f(x) = y$, and this is the maximum. □

Theorem (Berge's Theorem of the Maximum)**.** *Let $G : X \twoheadrightarrow Y$ be a continuous correspondence with non-empty compact values, and let $f : Gr(G) \to \mathbb{R}$ be continuous. Define the function $m : X \to \mathbb{R}$ as:*

$$m(x) = \max_{y \in G(x)} f(x, y),$$

and the correspondence $H : X \twoheadrightarrow Y$ of maximizers as:

$$H(x) = \{y \in G(x) | f(x, y) = m(x)\},$$

Then:

1. *m is continuous,*

2. *H is nonempty and compact for all x,*

3. *If Y is Hausdorff, then H is upper semicontinuous (according to both definitions).*

Make note of the assumptions here:

1. G is both upper and lower semicontinuous.

2. f is continuous.

Before proving this theorem, I will state and prove the following two lemmas:

Lemma 2. *Let $G : X \twoheadrightarrow Y$ be an LSC correspondence between topological spaces and let $f : Gr(G) \to \mathbb{R}$ be an LSC function. Define $m : X \to \bar{\mathbb{R}}$ by:*

$$m(x) = \sup_{y \in G(x)} f(x, y),$$

where $\sup \varnothing = -\infty$. Then m is a LSC function.

Proof. What needs to be shown is that $\{x \in X | m(x) > \alpha\}$ is open for all $\alpha \in \bar{\mathbb{R}}$. If $m(x_0) > \alpha$ for some $x_0 \in X$, then $f(x_0, y_0) > \alpha$ for some $y_0 \in G(x_0)$, which implies that $G(x_0)$ is nonempty. Because f is LSC, the set $W = \{(x, y) \in Gr(G) | f(x, y) > \alpha\}$ is an open neighborhood of (x_0, y_0). Therefore, there exist open neighborhoods U of x_0 and V of y_0 such that $(U \times V) \cap Gr(G) \subseteq W$. Note that $N = U \cap G^{-}(V)$ is an open neighborhood of x_0. For each $x \in N$, there exists a $y \in G(x) \cap V$ such that $(x, y) \in (U \times V) \cap Gr(G) \subseteq W$. Therefore, $f(x, y) > \alpha$, so $m(x) > \alpha$ for all $x \in N$, so $\{x \in X | m(x) > \alpha\}$ is open and therefore m is LSC. □

Lemma 3. *Let $G : X \twoheadrightarrow Y$ be a nonempty compact-valued USC correspondence between topological spaces and let $f : Gr(G) \to \mathbb{R}$ be a USC function. Define $m : X \to \bar{\mathbb{R}}$ by:*

$$m(x) = \max_{y \in G(x)} f(x, y),$$

Then m is a USC function.

Proof. Note that for each fixed x, $f(x, \cdot) : G(x) \to \mathbb{R}$ is well-defined and USC. Therefore, the maximum exists, so we can work with the maximum rather than the supremum. What needs to be shown is that $\{x \in X | m(x) < \alpha\}$ is open for all $\alpha \in \mathbb{R}$. Fix $\alpha \in \mathbb{R}$ such that this set is non-empty (otherwise it is trivially open) and let x_0 be such that $m(x_0) < \alpha$, and let $W = \{(x, y) \in Gr(G) | f(x, y) < \alpha\}$. Note that, for each $y \in G(x_0)$, it is the case that $(x_0, y) \in W$.

Since f is a USC function on $Gr(G)$, W is open. Therefore, for each $y \in G(x_0)$, there is an open neighborhood U_y containing x_0 and V_y containing y such that $(U_y \times V_y) \cap Gr(G) \subseteq W$. The family of V_y sets forms an open cover of W, and let $V = \cup_{i=1}^{n} V_{y_i}$ be a finite subcover, and let U be the corresponding set $\cup_{i=1}^{n} U_y$. Then $(U \times V) \cap Gr(G) \subseteq W$, and the USC of G guarantees that $N = U \cap G^{+}(V)$ is an open neighborhood of x_0. For each $x \in N$, if $y \in G(x)$, $f(x, y) < \alpha$, so $m(x) < \alpha$. □

Lemma 4. *The intersection of a SUSC correspondence and an OSUSC compact-valued correspondence is OSUSC and compact-valued.*

Now for the proof of the Theorem of the Maximum:

Proof. (To simplify the proof, I will make an additional assumption that X is a metric space (or even just a first-countable space)). The above lemmata imply that m is continuous. As the maximum exists, $H(x)$ is non-empty, and as f is USC, $H(x)$ is compact-valued. To show that $H(x)$ is a USC correspondence, it is sufficient to show that when $(x_n, y_n) \to (x, y)$ and $y_n \in H(x_n)$, then $y \in H(x)$. Note that $y \in G(x)$ because $G(x)$ is continuous, and because $\lim_{n\to\infty} f(x_n, y_n) = f(x, y)$, $y \in H(x)$. This implies that H is sequentially upper semicontinuous.

Using Lemma 4 and the fact that $H(x) \subseteq G(x)$ for all x, this implies that H is OSUSC. □

5.2 Fixed Point Theorems

Theorem (Intermediate Value)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) < c < f(b)$, then there exists an $x \in (a, b)$ such that $f(x) = c$.*

Proof. $[a, b]$ is connected, so $f([a, b])$ is connected. Therefore, for each $c \in f([a, b])$, there exists an x such that $f(x) = c$. □

Theorem ("Baby Brouwer")**.** *Let $[a, b] \subseteq \mathbb{R}$ and $f : [a, b] \to [a, b]$ be continuous. Then f has a fixed point.*

Proof. If $f(a) = a$ or $f(b) = b$, we are done. Otherwise, let $g(x) = x - f(x)$. Clearly $g(a) < 0$ and $g(b) > 0$, so there must exist an $x \in [a, b]$ such that $g(x) = 0$, which implies that $f(x) = x$ □

Theorem (Brouwer)**.** *Let $K \subseteq \mathbb{R}^M$ (with the standard topology) be a compact convex set, and let $f : K \to K$ be continuous. Then f has a fixed point.*

Given how simple it was to prove the "Baby Brouwer" Theorem, one may think that this theorem is simple to prove too. Unfortunately, that is not the case, and therefore we will skip this.

Theorem (Kakutani)**.** *Let* $K \subseteq \mathbb{R}^M$ *(with the standard topology) be a compact convex set, and let* $F : K \twoheadrightarrow K$ *be an upper semicontinuous compact-valued correspondence. Then* F *has a fixed point.*